Module II: Transformations

Computer Aided Design & Analysis

1. Matrix Representation of Points, Lines, and Planes

Points

- A point in 2D can be represented as a column vector:

$$\mathbf{P} = \begin{bmatrix} x \\ y \end{bmatrix}$$

- In 3D:

$$\mathbf{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Lines

- In 2D, a line with equation $ax + by + c = 0$ can be represented as the vector $[a, b, c]^T$.
- In 3D, a line may be defined parametrically using two points or a point and a direction vector.

Planes

- In 3D, a plane is represented as $ax + by + cz + d = 0$ or as the vector $[a, b, c, d]^T$.

2. 2D Transformations

2D geometric transformations alter the position, orientation, or size of shapes in a coordinate plane. They are typically represented using 3×3 matrices for ease of concatenation through homogeneous coordinates.

a. Translation

Moves a point by a specified distance in x and y.

Transformation Matrix:

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Applied as:
$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

T \cdot

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$

b. Scaling

Alters the size of an object relative to the origin.

$$S = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c. Rotation

Rotates a point by angle $ \theta $ about the origin.

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d. Reflection

Reflects a point over a specified axis.

- Over x-axis:

$$M_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Over y-axis:

$$M_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Homogeneous Representation & Concatenation

Homogeneous Coordinates

- Add an extra dimension to represent all affine transformations as matrix multiplication.
- For 2D: (x, y) → $(x, y, 1)$
- For 3D: (x, y, z) → $(x, y, z, 1)$

Concatenation (Composition)

- Transformations are combined by multiplying their matrices in sequence.
- If $ M_1 $, $ M_2 $, $ M_3 $ are matrices for translation, rotation, and scaling, the combined transformation is $ C = M_3 \cdot M_2 \cdot M_1 $.
- The order of multiplication matters (non-commutative).

4. 3D Transformations

3D transformations use $ 4 \times 4 $ matrices in homogeneous coordinates.

a. Translation

$$T = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b. Scaling

$$S = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c. Rotation

- About x-axis:

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- About y-axis:

$$R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- About z-axis:

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d. Reflection (over principal planes)

- Over x-y plane (z = 0):

$$\text{Reflect}_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

e. General 3D Transformation Concatenation

Multiple 3D transformations are combined by multiplying respective $ 4 \times 4 $ matrices, following the order of operations required by the application.

Summary Table: 2D and 3D Transformation Matrices

Transformation	2D (3×3) Matrix	3D (4×4) Matrix
Translation	$ T $	$ T $
Scaling	$ S $	$ S $
Rotation	$ R $	$ R_x, R_y, R_z $
Reflection	$ M_x, M_y $	Over principal planes
Homogeneous Form	Uses extra dimension (w = 1)	w = 1

Applications in CAD/CAM

- Precise geometric modeling and editing
- Animation and simulation of parts/assemblies
- Complex object transformation in graphics and manufacturing workflows

Understanding and utilizing these transformation matrices and concepts is fundamental for effective design, analysis, and visualization in computer-aided design and engineering.